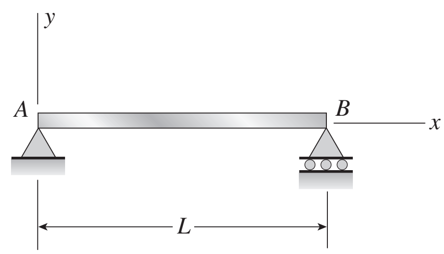
# 9

# Deflections of Beams

## Differential Equations of the Deflection Curve

*The beams described in the problems for Section 9.2 have constant flexural rigidity EI.*




Probs. 9.2-1 and 9.2-2

**Problem 9.2-1** The deflection curve for a simple beam *AB* (see figure) is given by the following equation:

$$v = -\frac{q_0 x}{360LEI}(7L^4 - 10L^2x^2 + 3x^4)$$

Describe the load acting on the beam.

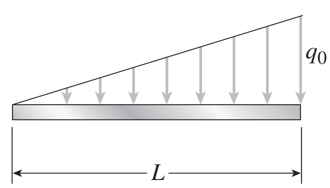
**Solution 9.2-1 Simple beam**

$$v = -\frac{q_0 x}{360\,LEI}(7L^4 - 10\,L^2x^2 + 3x^4)$$

Take four consecutive derivatives and obtain:

$$v'''' = -\frac{q_0 x}{LEI}$$

From Eq. (9-12c): $q = -EIv'''' = \frac{q_0 x}{L}$ ←




The load is a downward triangular load of maximum intensity $q_0$. ←

**Problem 9.2-2** The deflection curve for a simple beam *AB* (see figure) is given by the following equation:

$$v = -\frac{q_0 L^4}{\pi^4 EI} \sin\frac{\pi x}{L}$$

(a) Describe the load acting on the beam.
(b) Determine the reactions $R_A$ and $R_B$ at the supports.
(c) Determine the maximum bending moment $M_{max}$.

**Solution 9.2-2** **Simple beam**

$$v = -\frac{q_0 L^4}{\pi^4 EI}\sin\frac{\pi x}{L}$$

$$v' = -\frac{q_0 L^3}{\pi^3 EI}\cos\frac{\pi x}{L}$$

$$v'' = \frac{q_0 L^2}{\pi^2 EI}\sin\frac{\pi x}{L}$$

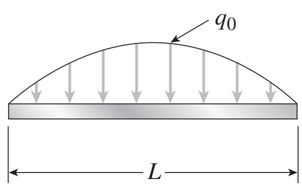
$$v''' = \frac{q_0 L}{\pi EI}\cos\frac{\pi x}{L}$$

$$v'''' = -\frac{q_0}{EI}\sin\frac{\pi x}{L}$$

(a) LOAD (EQ. 9-12c)

$$q = -EIv'''' = q_0 \sin\frac{\pi x}{L} \quad \longleftarrow$$

The load has the shape of a sine curve, acts downward, and has maximum intensity $q_0$. ⟵




(b) REACTIONS (EQ. 9-12b)

$$V = EIv''' = \frac{q_0 L}{\pi}\cos\frac{\pi x}{L}$$

At $x = 0$: $\quad V = R_A = \dfrac{q_0 L}{\pi} \quad \longleftarrow$

At $x = L$: $\quad V = -R_B = -\dfrac{q_0 L}{\pi};\ R_B = \dfrac{q_0 L}{\pi} \quad \longleftarrow$

(c) MAXIMUM BENDING MOMENT (EQ. 9-12a)

$$M = EIv'' = \frac{q_0 L^2}{\pi^2}\sin\frac{\pi x}{L}$$

For maximum moment, $x = \dfrac{L}{2};\ M_{max} = \dfrac{q_0 L^2}{\pi^2} \quad \longleftarrow$

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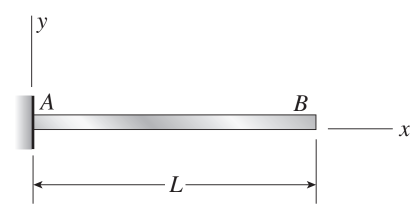
**Problem 9.2-3** The deflection curve for a cantilever beam *AB* (see figure) is given by the following equation:

$$v = -\frac{q_0 x^2}{120LEI}(10L^3 - 10L^2x + 5Lx^2 - x^3)$$

Describe the load acting on the beam.




**Probs. 9.2-3 and 9.2-4**

**Solution 9.2-3** **Cantilever beam**

$$v = -\frac{q_0 x^2}{120\,LEI}(10L^3 - 10L^2x + 5Lx^2 - x^3)$$

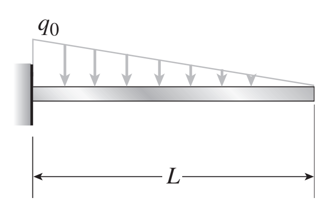
Take four consecutive derivatives and obtain:

$$v'''' = -\frac{q_0}{LEI}(L - x)$$

From Eq. (9-12c):

$$q = -EIv'''' = q_0\left(1 - \frac{x}{L}\right) \quad \longleftarrow$$

The load is a downward triangular load of maximum intensity $q_0$. ⟵



**Problem 9.2-4** The deflection curve for a cantilever beam *AB* (see figure) is given by the following equation:

$$v = -\frac{q_0 x^2}{360L^2EI}(45L^4 - 40L^3x + 15L^2x^2 - x^4)$$

(a) Describe the load acting on the beam.
(b) Determine the reactions $R_A$ and $M_A$ at the support.

**Solution 9.2-4 Cantilever beam**

$$v = -\frac{q_0 x^2}{360\,L^2EI}(45L^4 - 40\,L^3x + 15\,L^2x^2 - x^4)$$

$$v' = -\frac{q_0}{60\,L^2EI}(15\,L^4x - 20\,L^3x^2 + 10\,L^2x^3 - x^5)$$

$$v'' = -\frac{q_0}{12\,L^2EI}(3\,L^4 - 8\,L^3x + 6\,L^2x^2 - x^4)$$

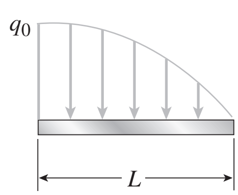
$$v''' = -\frac{q_0}{3\,L^2EI}(-2\,L^3 + 3\,L^2x - x^3)$$

$$v'''' = -\frac{q_0}{L^2EI}(L^2 - x^2)$$

(a) LOAD (EQ. 9-12c)

$$q = -EIv'''' = q_0\left(1 - \frac{x^2}{L^2}\right) \quad \leftarrow$$

The load is a downward parabolic load of maximum intensity $q_0$. ←




(b) REACTIONS $R_A$ AND $M_A$ (EQ. 9-12b AND EQ. 9-12a)

$$V = EIv''' = -\frac{q_0}{3L^2}(-2\,L^3 + 3\,L^2x - x^3)$$

At $x = 0$: $\quad V = R_A = \dfrac{2q_0L}{3} \quad \leftarrow$

$$M = EIv'' = -\frac{q_0}{12L^2}(3\,L^4 - 8\,L^3x + 6\,L^2x^2 - x^4)$$

At $x = 0$: $\quad M = M_A = -\dfrac{q_0L^2}{4} \quad \leftarrow$

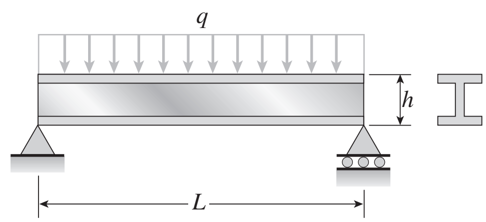
NOTE: Reaction $R_A$ is positive upward. Reaction $M_A$ is positive clockwise (minus means $M_A$ is counterclockwise).

## Deflection Formulas

*Problems 9.3-1 through 9.3-7 require the calculation of deflections using the formulas derived in Examples 9-1, 9-2, and 9-3. All beams have constant flexural rigidity EI.*

**Problem 9.3-1** A wide-flange beam (W 12 × 35) supports a uniform load on a simple span of length $L = 14$ ft (see figure).

Calculate the maximum deflection $\delta_{max}$ at the midpoint and the angles of rotation $\theta$ at the supports if $q = 1.8$ k/ft and $E = 30 \times 10^6$ psi. Use the formulas of Example 9-1.




**Probs. 9.3-1, 9.3-2 and 9.3-3**

**Solution 9.3-1 Simple beam (uniform load)**

W 12 × 35 $\quad L = 14 \text{ ft} = 168 \text{ in.}$

$q = 1.8 \text{ k/ft} = 150 \text{ lb/in.} \quad E = 30 \times 10^6 \text{ psi}$

$I = 285 \text{ in.}^4$

MAXIMUM DEFLECTION (EQ. 9-18)

$$\delta_{max} = \frac{5qL^4}{384EI} = \frac{5(150 \text{ lb/in.})(168 \text{ in.})^4}{384(30 \times 10^6 \text{ psi})(285 \text{ in.}^4)}$$

$$= 0.182 \text{ in.} \longleftarrow$$

ANGLE OF ROTATION AT THE SUPPORTS (EQS. 9-19 AND 9-20)

$$\theta = \theta_A = \theta_B = \frac{qL^3}{24EI} = \frac{(150 \text{ lb/in.})(168 \text{ in.})^3}{24(30 \times 10^6 \text{ psi})(285 \text{ in.}^4)}$$

$$= 0.003466 \text{ rad} = 0.199° \longleftarrow$$

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**Problem 9.3-2** A uniformly loaded steel wide-flange beam with simple supports (see figure) has a downward deflection of 10 mm at the midpoint and angles of rotation equal to 0.01 radians at the ends.

Calculate the height $h$ of the beam if the maximum bending stress is 90 MPa and the modulus of elasticity is 200 GPa. (*Hint:* Use the formulas of Example 9-1.)

**Solution 9.3-2 Simple beam (uniform load)**

$\delta = \delta_{max} = 10 \text{ mm} \quad \theta = \theta_A = \theta_B = 0.01 \text{ rad}$

$\sigma = \sigma_{max} = 90 \text{ MPa} \quad E = 200 \text{ GPa}$

Calculate the height $h$ of the beam.

Eq. (9-18): $\delta = \delta_{max} = \dfrac{5qL^4}{384EI}$ or $q = \dfrac{384EI\delta}{5L^4}$ (1)

Eq. (9-19): $\theta = \theta_A = \dfrac{qL^3}{24EI}$ or $q = \dfrac{24EI\theta}{L^3}$ (2)

Equate (1) and (2) and solve for $L$: $L = \dfrac{16\delta}{5\theta}$ (3)

Flexure formula: $\sigma = \dfrac{Mc}{I} = \dfrac{Mh}{2I}$

Maximum bending moment:

$M = \dfrac{qL^2}{8} \quad \therefore \sigma = \dfrac{qL^2h}{16I}$ (4)

Solve Eq. (4) for $h$: $h = \dfrac{16I\sigma}{qL^2}$ (5)

Substitute for $q$ from (2) and for $L$ from (3):

$$h = \frac{32\sigma\delta}{15E\theta^2} \longleftarrow$$

Substitute numerical values:

$$h = \frac{32(90 \text{ MPa})(10 \text{ mm})}{15(200 \text{ GPa})(0.01 \text{ rad})^2} = 96 \text{ mm} \longleftarrow$$

**Problem 9.3-3** What is the span length $L$ of a uniformly loaded simple beam of wide-flange cross section (see figure) if the maximum bending stress is 12,000 psi, the maximum deflection is 0.1 in., the height of the beam is 12 in., and the modulus of elasticity is $30 \times 10^6$ psi? (Use the formulas of Example 9-1.)

**Solution 9.3-3 Simple beam (uniform load)**

$\sigma = \sigma_{max} = 12{,}000$ psi $\quad \delta = \delta_{max} = 0.1$ in.
$h = 12$ in. $\quad E = 30 \times 10^6$ psi

Calculate the span length $L$.

Eq. (9-18): $\delta = \delta_{max} = \dfrac{5qL^4}{384\,EI}$ or $q = \dfrac{384\,EI\delta}{5L^4}$ (1)

Flexure formula: $\sigma = \dfrac{Mc}{I} = \dfrac{Mh}{2I}$

Maximum bending moment:

$M = \dfrac{qL^2}{8} \quad \therefore \sigma = \dfrac{qL^2h}{16I}$ (2)

Solve Eq. (2) for $q$: $q = \dfrac{16\,I\sigma}{L^2h}$ (3)

Equate (1) and (2) and solve for $L$:

$L^2 = \dfrac{24\,Eh\delta}{5\sigma} \quad L = \sqrt{\dfrac{24\,Eh\delta}{5\sigma}}$ ←

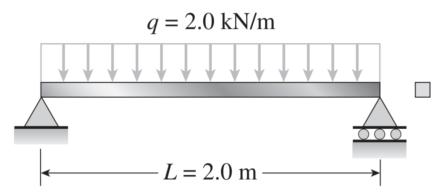
Substitute numerical values:

$$L^2 = \frac{24(30 \times 10^6 \text{ psi})(12 \text{ in.})(0.1 \text{ in.})}{5(12{,}000 \text{ psi})} = 14{,}400 \text{ in.}^2$$

$L = 120$ in. $= 10$ ft ←

**Problem 9.3-4** Calculate the maximum deflection $\delta_{max}$ of a uniformly loaded simple beam (see figure) if the span length $L = 2.0$ m, the intensity of the uniform load $q = 2.0$ kN/m, and the maximum bending stress $\sigma = 60$ MPa.

The cross section of the beam is square, and the material is aluminum having modulus of elasticity $E = 70$ GPa. (Use the formulas of Example 9-1.)




**Solution 9.3-4 Simple beam (uniform load)**

$L = 2.0$ m $\quad q = 2.0$ kN/m
$\sigma = \sigma_{max} = 60$ MPa $\quad E = 70$ GPa

CROSS SECTION (square; $b$ = width)

$I = \dfrac{b^4}{12} \quad S = \dfrac{b^3}{6}$

Maximum deflection (Eq. 9-18): $\delta = \dfrac{5qL^4}{384\,EI}$ (1)

Substitute for $I$: $\delta = \dfrac{5qL^4}{32\,Eb^4}$ (2)

Flexure formula with $M = \dfrac{qL^2}{8}$: $\quad \sigma = \dfrac{M}{S} = \dfrac{qL^2}{8S}$

Substitute for $S$: $\sigma = \dfrac{3qL^2}{4b^3}$ (3)

Solve for $b^3$: $b^3 = \dfrac{3qL^2}{4\sigma}$ (4)

Substitute $b$ into Eq. (2): $\delta_{max} = \dfrac{5L\sigma}{24E}\left(\dfrac{4L\sigma}{3q}\right)^{1/3}$ ←

(The term in parentheses is nondimensional.)

Substitute numerical values:

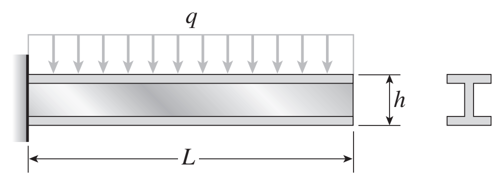
$$\frac{5L\sigma}{24E} = \frac{5(2.0 \text{ m})(60 \text{ MPa})}{24(70 \text{ GPa})} = \frac{1}{2800} \text{ m} = \frac{1}{2.8} \text{ mm}$$

$$\left(\frac{4\,L\sigma}{3q}\right)^{1/3} = \left[\frac{4(2.0 \text{ m})(60 \text{ MPa})}{3(2000 \text{ N/m})}\right]^{1/3} = 10(80)^{1/3}$$

$$\delta_{max} = \frac{10(80)^{1/3}}{2.8} \text{ mm} = 15.4 \text{ mm}$$ ←

**Problem 9.3-5** A cantilever beam with a uniform load (see figure) has a height $h$ equal to 1/8 of the length $L$. The beam is a steel wide-flange section with $E = 28 \times 10^6$ psi and an allowable bending stress of 17,500 psi in both tension and compression.

Calculate the ratio $\delta/L$ of the deflection at the free end to the length, assuming that the beam carries the maximum allowable load. (Use the formulas of Example 9-2.)




**Solution 9.3-5 Cantilever beam (uniform load)**

$$\frac{h}{L} = \frac{1}{8} \qquad E = 28 \times 10^6 \text{ psi} \qquad \sigma = 17{,}500 \text{ psi}$$

Calculate the ratio $\delta/L$.

Maximum deflection (Eq. 9-26): $$\delta_{max} = \frac{qL^4}{8\,EI} \qquad (1)$$

$$\therefore \quad \frac{\delta}{L} = \frac{qL^3}{8EI} \qquad (2)$$

Flexure formula with $M = \dfrac{qL^2}{2}$:

$$\sigma = \frac{Mc}{I} = \left(\frac{qL^2}{2}\right)\left(\frac{h}{2I}\right) = \frac{qL^2h}{4I}$$

Solve for $q$:

$$q = \frac{4I\sigma}{L^2h} \qquad (3)$$

Substitute $q$ from (3) into (2):

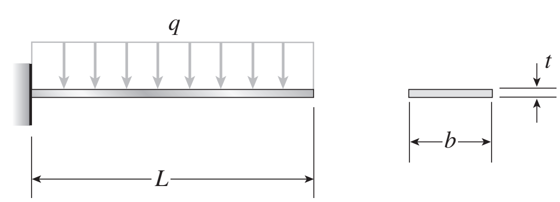
$$\frac{\delta}{L} = \frac{\sigma}{2E}\left(\frac{L}{h}\right) \quad \longleftarrow$$

Substitute numerical values:

$$\frac{\delta}{L} = \frac{17{,}500 \text{ psi}}{2(28 \times 10^6 \text{ psi})}(8) = \frac{1}{400} \quad \longleftarrow$$

**Problem 9.3-6** A gold-alloy microbeam attached to a silicon wafer behaves like a cantilever beam subjected to a uniform load (see figure). The beam has length $L = 27.5$ μm and rectangular cross section of width $b = 4.0$ μm and thickness $t = 0.88$ μm. The total load on the beam is 17.2 μN.

If the deflection at the end of the beam is 2.46 μm, what is the modulus of elasticity $E_g$ of the gold alloy? (Use the formulas of Example 9-2.)




**Solution 9.3-6 Gold-alloy microbeam**

Cantilever beam with a uniform load.

$L = 27.5\ \mu\text{m} \qquad b = 4.0\ \mu\text{m} \qquad t = 0.88\ \mu\text{m}$

$qL = 17.2\ \mu\text{N} \qquad \delta_{max} = 2.46\ \mu\text{m}$

Determine $E_g$.

Eq. (9-26): $$\delta = \frac{qL^4}{8E_gI} \quad \text{or} \quad E_g = \frac{qL^4}{8I\delta_{max}}$$

$$I = \frac{bt^3}{12} \qquad E_g = \frac{3\,qL^4}{2\,bt^3\delta_{max}} \quad \longleftarrow$$

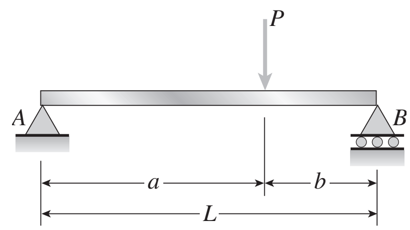
Substitute numerical values:

$$E_g = \frac{3(17.2\ \mu\text{N})(27.5\ \mu\text{m})^3}{2(4.0\ \mu\text{m})(0.88\ \mu\text{m})^3(2.46\ \mu\text{m})}$$

$$= 80.02 \times 10^9 \text{ N/m}^2 \quad \text{or} \quad E_g = 80.0 \text{ GPa} \quad \longleftarrow$$

**Problem 9.3-7** Obtain a formula for the ratio $\delta_C/\delta_{max}$ of the deflection at the midpoint to the maximum deflection for a simple beam supporting a concentrated load $P$ (see figure).

From the formula, plot a graph of $\delta_C/\delta_{max}$ versus the ratio $a/L$ that defines the position of the load ($0.5 < a/L < 1$). What conclusion do you draw from the graph? (Use the formulas of Example 9-3.)




**Solution 9.3-7** **Simple beam (concentrated load)**

Eq. (9-35): $\delta_C = \dfrac{Pb(3L^2 - 4b^2)}{48EI} \quad (a \geq b)$

Eq. (9-34): $\delta_{max} = \dfrac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}\,LEI} \quad (a \geq b)$

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3}L)(3L^2 - 4b^2)}{16(L^2 - b^2)^{3/2}} \quad (a \geq b)$$

Replace the distance $b$ by the distance $a$ by substituting $L - a$ for $b$:

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3}L)(-L^2 + 8aL - 4a^2)}{16(2aL - a^2)^{3/2}}$$

Divide numerator and denominator by $L^2$:

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3}L)\left(-1 + 8\dfrac{a}{L} - 4\dfrac{a^2}{L^2}\right)}{16L\left(2\dfrac{a}{L} - \dfrac{a^2}{L^2}\right)^{3/2}}$$

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3})\left(-1 + 8\dfrac{a}{L} - 4\dfrac{a^2}{L^2}\right)}{16\left(2\dfrac{a}{L} - \dfrac{a^2}{L^2}\right)^{3/2}} \quad \longleftarrow$$

Alternative form of the ratio

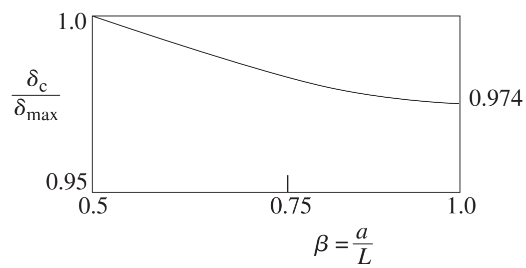
Let $\beta = \dfrac{a}{L}$

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3})(-1 + 8\beta - 4\beta^2)}{16(2\beta - \beta^2)^{3/2}} \quad \longleftarrow$$

Graph of $\delta_c/\delta_{max}$ versus $\beta = a/L$

Because $a \geq b$, the ratio $\beta$ versus from 0.5 to 1.0.

| $\beta$ | $\dfrac{\delta_c}{\delta_{max}}$ |
|---|---|
| 0.5 | 1.0 |
| 0.6 | 0.996 |
| 0.7 | 0.988 |
| 0.8 | 0.981 |
| 0.9 | 0.976 |
| 1.0 | 0.974 |

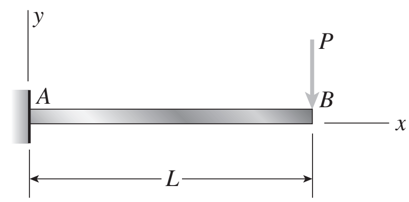



Note: The deflection $\delta_c$ at the midpoint of the beam is almost as large as the maximum deflection $\delta_{max}$. The greatest difference is only 2.6% and occurs when the load reaches the end of the beam ($\beta = 1$).

## Deflections by Integration of the Bending-Moment Equation

*Problems 9.3-8 through 9.3-16 are to be solved by integrating the second-order differential equation of the deflection curve (the bending-moment equation). The origin of coordinates is at the left-hand end of each beam, and all beams have constant flexural rigidity EI.*

**Problem 9.3-8** Derive the equation of the deflection curve for a cantilever beam $AB$ supporting a load $P$ at the free end (see figure). Also, determine the deflection $\delta_B$ and angle of rotation $\theta_B$ at the free end. (*Note:* Use the second-order differential equation of the deflection curve.)




**Solution 9.3-8 Cantilever beam (concentrated load)**

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -P(L - x)$$

$$EIv' = -PLx + \frac{Px^2}{2} + C_1$$

B.C. $v'(0) = 0 \quad \therefore C_1 = 0$

$$EIv = -\frac{PLx^2}{2} + \frac{Px^3}{6} + C_2$$

B.C. $v(0) = 0 \quad \therefore C_2 = 0$

$$v = -\frac{Px^2}{6EI}(3L - x) \longleftarrow$$

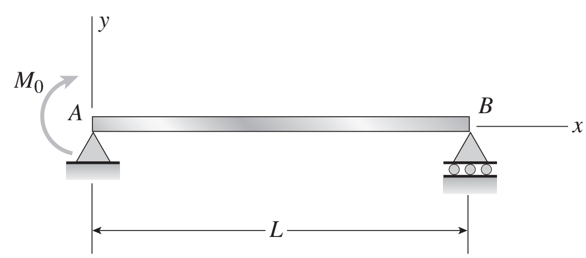
$$v' = -\frac{Px}{2EI}(2L - x)$$

$$\delta_B = -v(L) = \frac{PL^3}{3EI} \longleftarrow$$

$$\theta_B = -v'(L) = \frac{PL^2}{2EI} \longleftarrow$$

(These results agree with Case 4, Table G-1.)

**Problem 9.3-9** Derive the equation of the deflection curve for a simple beam $AB$ loaded by a couple $M_0$ at the left-hand support (see figure). Also, determine the maximum deflection $\delta_{max}$. (*Note:* Use the second-order differential equation of the deflection curve.)




**Solution 9.3-9 Simple beam (couple $M_0$)**

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = M_0\left(1 - \frac{x}{L}\right)$$

$$EIv' = M_0\left(x - \frac{x^2}{2L}\right) + C_1$$

$$EIv = M_0\left(\frac{x^2}{2} - \frac{x^3}{6L}\right) + C_1x + C_2$$

B.C. $v(0) = 0 \quad \therefore C_2 = 0$

B.C. $v(L) = 0 \quad \therefore C_1 = -\frac{M_0L}{3}$

$$v = -\frac{M_0x}{6\,LEI}(2L^2 - 3Lx + x^2) \longleftarrow$$

MAXIMUM DEFLECTION

$$v' = -\frac{M_0}{6\,LEI}(2\,L^2 - 6\,Lx + 3\,x^2)$$

Set $v' = 0$ and solve for $x$:

$$x_1 = L\left(1 - \frac{\sqrt{3}}{3}\right) \longleftarrow$$

Substitute $x_1$ into the equation for $v$:

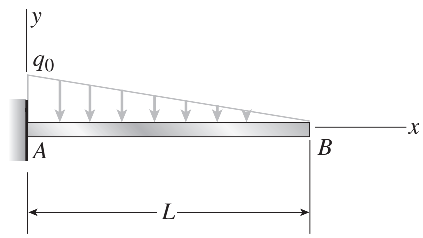
$$\delta_{max} = -(v)_{x-x_1}$$

$$= \frac{M_0L^2}{9\sqrt{3}EI} \longleftarrow$$

(These results agree with Case 7, Table G-2.)

**Problem 9.3-10** A cantilever beam $AB$ supporting a triangularly distributed load of maximum intensity $q_0$ is shown in the figure.

Derive the equation of the deflection curve and then obtain formulas for the deflection $\delta_B$ and angle of rotation $\theta_B$ at the free end. (*Note:* Use the second-order differential equation of the deflection curve.)




**Solution 9.3-10 Cantilever beam (triangular load)**

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -\frac{q_0}{6L}(L - x)^3$$

$$EIv' = \frac{q_0}{24L}(L - x)^4 + C_1$$

B.C. $v'(0) = 0 \quad \therefore C_1 = -\frac{q_0 L^3}{24}$

$$EIv = -\frac{q_0}{120L}(L - x)^5 - \frac{q_0 L^3 x}{24} + C_2$$

B.C. $v(0) = 0 \quad \therefore C_2 = \frac{q_0 L^4}{120}$

$$v = -\frac{q_0 x^2}{120\,LEI}(10L^3 - 10L^2 x + 5Lx^2 - x^3) \quad \longleftarrow$$

$$v' = -\frac{q_0 x}{24\,LEI}(4L^3 - 6L^2 x + 4Lx^2 - x^3)$$

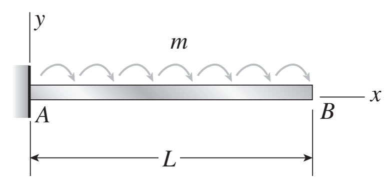
$$\delta_B = -v(L) = \frac{q_0 L^4}{30\,EI} \quad \longleftarrow$$

$$\theta_B = -v'(L) = \frac{q_0 L^3}{24\,EI} \quad \longleftarrow$$

(These results agree with Case 8, Table G-1.)

**Problem 9.3-11** A cantilever beam $AB$ is acted upon by a uniformly distributed moment (bending moment, not torque) of intensity $m$ per unit distance along the axis of the beam (see figure).

Derive the equation of the deflection curve and then obtain formulas for the deflection $\delta_B$ and angle of rotation $\theta_B$ at the free end. (*Note:* Use the second-order differential equation of the deflection curve.)




**Solution 9.3-11 Cantilever beam (distributed moment)**

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -m(L - x)$$

$$EIv' = -m\left(Lx - \frac{x^2}{2}\right) + C_1$$

B.C. $v'(0) = 0 \quad \therefore C_1 = 0$

$$EIv = -m\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2$$

B.C. $v(0) = 0 \quad \therefore C_2 = 0$

$$v = -\frac{mx^2}{6\,EI}(3L - x) \quad \longleftarrow$$

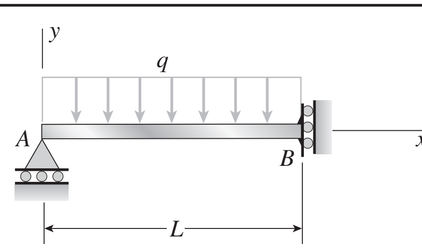
$$v' = -\frac{mx}{2EI}(2L - x)$$

$$\delta_B = -v(L) = \frac{mL^3}{3\,EI} \quad \longleftarrow$$

$$\theta_B = -v'(L) = \frac{mL^2}{2\,EI} \quad \longleftarrow$$

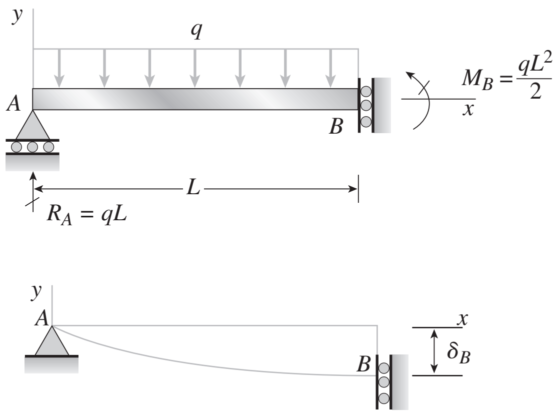
**Problem 9.3-12** The beam shown in the figure has a roller support at $A$ and a guided support at $B$. The guided support permits vertical movement but no rotation.

Derive the equation of the deflection curve and determine the deflection $\delta_B$ at end $B$ due to the uniform load of intensity $q$. (*Note:* Use the second-order differential equation of the deflection curve.)



**Solution 9.3-12 Beam with a guided support**

REACTIONS AND DEFLECTION CURVE




BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = qLx - \frac{qx^2}{2}$$

$$EIv' = \frac{qLx^2}{2} - \frac{qx^3}{6} + C_1$$

B.C. $v(L) = 0 \quad \therefore\ C_1 = -\frac{qL^3}{3}$

$$EIv = \frac{qLx^3}{6} - \frac{qx^4}{24} - \frac{qL^3x}{3} + C_2$$

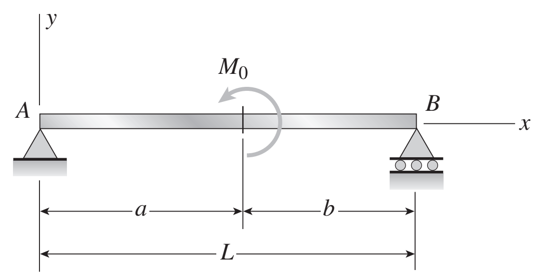
B.C. $v(0) = 0 \quad \therefore C_2 = 0$

$$v = -\frac{qx}{24EI}(8L^3 - 4Lx^2 + x^3) \quad \longleftarrow$$

$$\delta_B = -v(L) = \frac{5qL^4}{24EI} \quad \longleftarrow$$

---

**Problem 9.3-13** Derive the equations of the deflection curve for a simple beam $AB$ loaded by a couple $M_0$ acting at distance $a$ from the left-hand support (see figure). Also, determine the deflection $\delta_0$ at the point where the load is applied. (*Note:* Use the second-order differential equation of the deflection curve.)

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**Solution 9.3-13 Simple beam (couple $M_0$)**

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = \frac{M_0x}{L} \quad (0 \le x \le a)$$

$$EIv' = \frac{M_0x^2}{2L} + C_1 \quad (0 \le x \le a)$$

$$EIv'' = M = -\frac{M_0}{L}(L - x) \quad (a \le x \le L)$$

$$EIv' = -\frac{M_0}{L}\left(Lx - \frac{x^2}{2}\right) + C_2 \quad (a \le x \le L)$$

B.C. 1 $(v')_{\text{Left}} = (v')_{\text{Right}}$ at $x = a$

$\therefore C_2 = C_1 + M_0a$

$$EIv = \frac{M_0x^3}{6L} + C_1x + C_3 \quad (0 \le x \le a)$$

B.C. 2 $v(0) = 0 \quad \therefore C_3 = 0$

$$EIv = -\frac{M_0x^2}{2} + \frac{M_0x^3}{6L} + C_1x + M_0ax + C_4 \quad (a \le x \le L)$$

B.C. 3 $v(L) = 0 \quad \therefore C_4 = -M_0L\left(a - \frac{L}{3}\right) - C_1L$

B.C. 4 $(v)_{\text{Left}} = (v)_{\text{Right}}$ at $x = a$

$$\therefore C_4 = -\frac{M_0a^2}{2}$$

$$C_1 = \frac{M_0}{6L}(2L^2 - 6aL + 3a^2)$$

$$v = -\frac{M_0x}{6LEI}(6aL - 3a^2 - 2L^2 - x^2) \quad (0 \le x \le a) \quad \longleftarrow$$

$$v = -\frac{M_0}{6LEI}(3a^2L - 3a^2x - 2L^2x + 3Lx^2 - x^3) \quad \longleftarrow$$

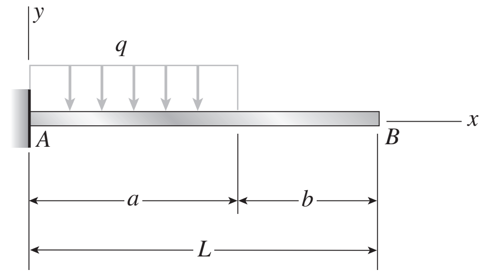
$(a \le x \le L)$

$$\delta_0 = -v(a) = \frac{M_0a(L - a)(2a - L)}{3LEI}$$

$$= \frac{M_0ab(2a - L)}{3LEI} \quad \longleftarrow$$

NOTE: $\delta_0$ is positive downward. The preceding results agree with Case 9, Table G-2.

**Problem 9.3-14** Derive the equations of the deflection curve for a cantilever beam $AB$ carrying a uniform load of intensity $q$ over part of the span (see figure). Also, determine the deflection $\delta_B$ at the end of the beam. (*Note:* Use the second-order differential equation of the deflection curve.)




**Solution 9.3-14 Cantilever beam (partial uniform load)**

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -\frac{q}{2}(a-x)^2 = -\frac{q}{2}(a^2 - 2ax + x^2) \quad (0 \le x \le a)$$

$$EIv' = -\frac{q}{2}\left(a^2x - ax^2 + \frac{x^3}{3}\right) + C_1 \quad (0 \le x \le a)$$

B.C. 1 $v'(0) = 0 \quad \therefore C_1 = 0$

$EIv'' = M = 0 \quad (a \le x \le L)$

$EIv' = C_2 \quad (a \le x \le L)$

B.C. 2 $(v')_{\text{Left}} = (v')_{\text{Right}}$ at $x = a$

$$\therefore C_2 = -\frac{qa^3}{6}$$

$$EIv = -\frac{q}{2}\left(\frac{a^2x^2}{2} - \frac{ax^3}{3} + \frac{x^4}{12}\right) + C_3 \quad (0 \le x \le a)$$

B.C. 3 $v(0) = 0 \quad \therefore C_3 = 0$

$$EIv = C_2x + C_4 = -\frac{qa^3x}{6} + C_4 \quad (a \le x \le L)$$

B.C. 4 $(v)_{\text{Left}} = (v)_{\text{Right}}$ at $x = a$

$$\therefore C_4 = \frac{qa^4}{24}$$

$$v = -\frac{qx^2}{24EI}(6a^2 - 4ax + x^2) \quad (0 \le x \le a) \quad \longleftarrow$$

$$v = -\frac{qa^3}{24EI}(4x - a) \quad (a \le x \le L) \quad \longleftarrow$$

$$\delta_B = -v(L) = \frac{qa^3}{24EI}(4L - a) \quad \longleftarrow$$

(These results agree with Case 2, Table G-1.)

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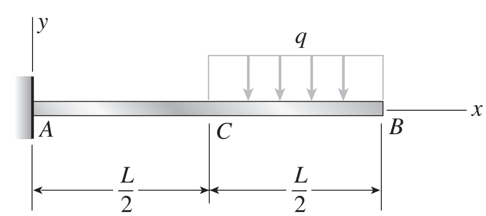
**Problem 9.3-15** Derive the equations of the deflection curve for a cantilever beam $AB$ supporting a uniform load of intensity $q$ acting over one-half of the length (see figure). Also, obtain formulas for the deflections $\delta_B$ and $\delta_C$ at points $B$ and $C$, respectively. (*Note:* Use the second-order differential equation of the deflection curve.)




**Solution 9.3-15 Cantilever beam (partial uniform load)**

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -\frac{qL}{8}(3L - 4x) \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$EIv' = -\frac{qL}{8}(3Lx - 2x^2) + C_1 \quad \left(0 \le x \le \frac{L}{2}\right)$$

B.C. 1 $v'(0) = 0 \quad \therefore C_1 = 0$

$$EIv'' = M = -\frac{q}{2}(L^2 - 2Lx + x^2) \quad \left(\frac{L}{2} \le x \le L\right)$$

$$EIv' = -\frac{q}{2}\left(L^2x - Lx^2 + \frac{x^3}{3}\right) + C_2 \quad \left(\frac{L}{2} \le x \le L\right)$$

*(Continued)*

B.C. 2 $(v')_{\text{Left}} = (v')_{\text{Right}}$ at $x = \dfrac{L}{2}$

$$\therefore\ C_2 = \frac{qL^3}{48}$$

$$EIv = -\frac{qL}{8}\left(\frac{3Lx^2}{2} - \frac{2x^3}{3}\right) + C_3 \quad \left(0 \le x \le \frac{L}{2}\right)$$

B.C. 3 $v(0) = 0 \quad \therefore\ C_3 = 0$

$$EIv = -\frac{q}{2}\left(\frac{L^2x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12}\right) + \frac{qL^3}{48}x + C_4 \quad \left(\frac{L}{2} \le x \le L\right)$$

B.C. 4 $(v)_{\text{Left}} = (v)_{\text{Right}}$ at $x = \dfrac{L}{2}$

$$\therefore\ C_4 = -\frac{qL^4}{384}$$

$$v = -\frac{qLx^2}{48\,EI}(9L - 4x) \quad \left(0 \le x \le \frac{L}{2}\right) \quad \longleftarrow$$

$$\delta_C = -v\left(\frac{L}{2}\right) = \frac{7qL^4}{192EI} \quad \longleftarrow$$

$$v = -\frac{q}{384\,EI}(16x^4 - 64\,Lx^3 + 96\,L^2x^2 - 8\,L^3x + L^4) \quad \left(\frac{L}{2} \le x \le L\right) \quad \longleftarrow$$

$$\delta_B = -v(L) = \frac{41qL^4}{384EI} \quad \longleftarrow$$

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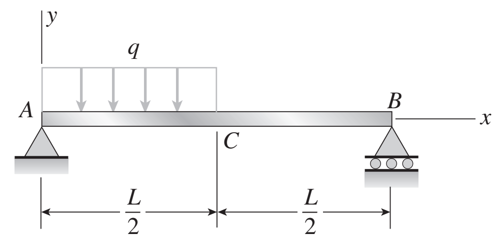
**Problem 9.3-16** Derive the equations of the deflection curve for a simple beam $AB$ with a uniform load of intensity $q$ acting over the left-hand half of the span (see figure). Also, determine the deflection $\delta_C$ at the midpoint of the beam. (*Note:* Use the second-order differential equation of the deflection curve.)




**Solution 9.3-16** **Simple beam (partial uniform load)**

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = \frac{3qLx}{8} - \frac{qx^2}{2} \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$EIv' = \frac{3qLx^2}{16} - \frac{qx^3}{6} + C_1 \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$EIv'' = M = \frac{qL^2}{8} - \frac{qLx}{8} \quad \left(\frac{L}{2} \le x \le L\right)$$

$$EIv' = \frac{qL^2x}{8} - \frac{qLx^2}{16} + C_2 \quad \left(\frac{L}{2} \le x \le L\right)$$

B.C. 1 $(v')_{\text{Left}} = (v')_{\text{Right}}$ at $x = \dfrac{L}{2}$

$$\therefore\ C_2 = C_1 - \frac{qL^3}{48}$$

$$EIv = \frac{qLx^3}{16} - \frac{qx^4}{24} + C_1x + C_3 \quad \left(0 \le x \le \frac{L}{2}\right)$$

B.C. 2 $v(0) = 0 \quad \therefore\ C_3 = 0$

$$EIv = \frac{qL^2x^2}{16} - \frac{qLx^3}{48} + C_1x - \frac{qL^3x}{48} + C_4 \quad \left(\frac{L}{2} \le x \le L\right)$$

B.C. 3 $v(L) = 0 \quad \therefore\ C_4 = -C_1L - \dfrac{qL^4}{48}$

B.C. 4 $(v)_{\text{Left}} = (v)_{\text{Right}}$ at $x = \dfrac{L}{2}$

$$\therefore\ C_1 = -\frac{3qL^3}{128}$$

$$v = -\frac{qx}{384EI}(9L^3 - 24Lx^2 + 16x^3) \quad \left(0 \le x \le \frac{L}{2}\right) \quad \longleftarrow$$

$$v = -\frac{qL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3) \quad \longleftarrow \quad \left(\frac{L}{2} \le x \le L\right)$$

$$\delta_C = -v\left(\frac{L}{2}\right) = \frac{5qL^4}{768EI} \quad \longleftarrow$$

(These results agree with Case 2, Table G-2.)